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Flow near the leading edge of a rectangular wing of small aspect ratio with applications to the bow of a ship*

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SUMMARY

A local solution is developed for the square-root singularity at the leading edge of a rectangular flat plate of small aspect ratio. The analysis is based on the assumption that the spanwise load distribution is elliptical, and it follows that the two-dimensional lifting-surface integral equation can be reduced to a one-dimensional integral equation of the Wiener–Hopf form. A solution is then obtained, valid near the leading edge, which yields the strength of the square-root singularity. This solution is used to compute the leading-edge suction force, which differs by 15% from the known value based on the total lift and drag expressions. The local solution is applied to study the flow around the stem of a ship's bow, as a result of a yawing motion, using Lighthill's rule to correct for the finite radius of curvature at the bow.

1. Introduction

Local cavitation can occur near the bow of a high-speed ship as a result of straight-ahead motion, if the bow is of sufficiently fine form and the stem is not carefully designed. It is also possible for cavitation or ventilation to occur as a result of oblique inflow angles, either during a turning maneuver of the ship or as a result of oscillatory lateral motions in a seaway. The case of straight-ahead flow can be analyzed with a source distribution, in the same manner as the thickness problem of thin-wing theory, but the oblique-inflow velocity distribution is more difficult to predict.

If the Froude number is not large, one can ignore free-surface effects, replacing the free surface by a rigid flat plane and reflecting about this boundary with a simple image solution. Thus the ship's hull becomes, in effect, a low-aspect-ratio lifting surface, and the flow near the bow of the ship is identical to the flow near the leading edge of the lifting surface. Since ships' bows are predominantly vertical, the resulting equivalent lifting surface is a rectangular flat plate, and it is well known that low-aspect-ratio wing theory is invalid near the leading edge, especially when the planform is rectangular. Thus we shall focus our attention here on the leading-edge singularity of a rectangular flat plate of small aspect ratio and, once the strength of this square-root singularity has been obtained, use Lighthill's rule to obtain the corresponding surface velocity near a leading edge with small but finite radius of curvature.

A closely related study was carried out by Wieghardt [8], on the chordwise load distribution of a series of rectangular wings of various aspect ratios. Numerical solutions were utilized for this purpose, but it was found that the procedure became inefficient as the aspect ratio approached zero. For this case, and based on the fundamental assumption that the spanwise loading is elliptical, Wieghardt derived a one-dimensional integral equation, equivalent to that which we shall study here. He obtained approximate solutions to this equation with numerical techniques and, in fact, found the correct form for the leading-edge singularity, with a coefficient which is numerically identical to that which we obtain. Our method is based instead on the use of the Wiener–Hopf technique, since the one-dimensional integral equation referred to above is of the Wiener–Hopf form. Indeed, this integral equation is similar to Prandtl's lifting-line

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integral equation, but with a semi-infinite range of integration and a more complicated kernel function. The technique which we shall utilize is similar to that employed by Stewartson [4], who analyzed the local behavior near the tip of a rectangular foil of high aspect ratio, assuming the latter to be governed by the lifting-line equation. (Stewartson's approach is criticized by Van Dyke [5], on the grounds that the lifting-line equation is invalid near the tip; nevertheless, we emphasize that Stewartson's mathematical approach is appropriate to the present problem.)

The Wiener–Hopf integral equation which we must solve involves in its kernel a complete elliptic integral of the second kind, and an exact factorization of this kernel appears to be impossible. However, we focus our attention here on the local solution near the leading edge, and hence on the asymptotic form of the Fourier-transformed kernel for large values of the transform variable. In this manner an symptotic approximation can be found for the factorized kernel, which permits us to compute the strength of the singularity near the leading edge.

Aside from the usual assumptions of linearized thin-wing theory, and an incompressible fluid, we make the two additional and important assumptions that free-surface effects are not important, and that the spanwise loading (i.e., vertical distribution of vorticity, in the reference frame of a ship) is elliptical. In principle, both assumptions can be avoided if a numerical solution is sought, based on the two-dimensional integral equation which governs the lifting problem of a yawed thin ship in steady motion on the free surface. Such a task is of considerable difficulty, however, if indeed it is within the ability of even a numerical solution. Thus the present results must be regarded as a first step toward the complete treatment of this problem.

2. The lifting-surface equation

We consider a rectangular flat plate, of span $2s$ and length L. Cartesian coordinates are chosen with the plate lying in the x–y plane, the leading edge at $x=0$, trailing edge at $x=L$, and tips at $y=\pm s$. The inflow velocity vector is $(U, 0, U\alpha)$, where $\alpha \ll 1$ is the angle of attack. The lifting-surface integral equation for this problem is derived by Robinson and Laurmann [3], in the form

$$U\alpha = \frac{1}{4\pi}\int_{-s}^{s} d\eta \int_{0}^{L} d\xi \, \frac{\partial^2 \tau}{\partial\xi\partial\eta}\left\{\frac{[(x-\xi)^2+(y-\eta)^2]^{\frac{1}{2}}}{(x-\xi)(y-\eta)} + \frac{1}{y-\eta}\right\}, 0< x < l, \quad |y| < s \tag{1}$$

where the singularities are to be interpreted in the Cauchy principle-value sense. The unknown $\tau(\xi, \eta)$ is the moment of normal dipoles, distributed over the lifting surface, and satisfies the condition $\tau=0$ on the leading edge and tips. Since this dipole moment is equal to the jump in velocity potential ϕ between the two sides of the plate, and it is odd in y, it follows that

$$\phi\,(x, y, 0\pm) = \tfrac{1}{2}\tau(x, y)\,. \tag{2}$$

We are concerned here with the approximate solution of (1) for the case where the aspect ratio $2s/L$ is small. The usual approximation is obtained by disregarding the vicinity of the leading and trailing edges: if, over the substantial portion of the foil, $|x-\xi| \gg |y-\eta|$, the kernel of (1) can be simplified to

$$\{\quad\} \cong \frac{1}{y-\eta}\,[1+\mathrm{sgn}\,(x-\xi)]\,. \tag{3}$$

The ξ-integration can then be performed, and we obtain the integral equation

$$U\alpha = \frac{1}{2\pi}\int_{-s}^{s} \frac{d\eta}{y-\eta}\,\frac{\partial\tau(x, \eta)}{\partial\eta}, \qquad |y|< s \tag{4}$$

The appropriate solution of this equation is the elliptic load distribution

$$\tau = 2U\alpha(s^2-\eta^2)^{\frac{1}{2}}\,. \tag{5}$$

This is the familiar solution for the two-dimensional cross-flow, with velocity $U\alpha$, past a flat plate, and it is valid in a stripwise sense, for the three-dimensional foil, except in the vicinity of the leading edge. Formally, the first-order solution of (1), if the aspect ratio is small, is

$$\tau(x, y) = 2U\alpha H(x)(s^2 - y^2)^{\frac{1}{2}} \tag{6}$$

where $H(x)$ denotes the Heaviside unit step function, $H(x)=0$ if $x<0$, and $H(x)=1$ if $x>0$.

Near the leading edge, for $|x| \ll s$, the solution corresponding to (6) will appear as a bound vortex, of elliptic strength, and a suitable distribution of trailing vortices. Indeed (6) is equivalent to that vortex system, everywhere in the plate (and wake), by an application of Stokes' theorem. But we know that the flow in the vicinity of the leading edge must instead be of the form

$$\tau(x, y) = U\alpha C(y) x^{\frac{1}{2}} \qquad |x| \ll s \tag{7}$$

i.e., the local flow is similar to a plane flow, in the x–z plane, past a semi-infinite flat plate.

It should be noted that the limiting form (7) for the dipole moment cannot be obtained directly from (1) by a stretching argument along the leading edge, say $|x-\xi| \ll y-\eta$. This procedure leads to the approximate kernel

$$\{ \quad \} \cong \frac{1}{x-\xi} \operatorname{sgn}(y-\eta) \tag{8}$$

and hence to the integral equation

$$U\alpha = \frac{1}{4\pi} \int_0^\infty d\xi \frac{\partial \tau}{\partial \xi} \frac{1}{x-\xi}, \qquad x>0 \tag{9}$$

The square-root function (7) is a homogeneous solution of equation (9), with the left side of (9) replaced by zero, and thus (9) cannot help us to find the function $C(y)$ in (7), or the strength of the leading-edge singularity. Physically, it is not surprising that this procedure fails, for the first-order solution (6) is valid only when $x \gg s$, and the local solution (7) when $x \ll s$. Thus, we cannot expect a simple matching of these two regimes, and it is necessary to employ the full generality of the lifting-surface integral equation (1) in the region $x = O(s)$.

If solutions of (1) are sought, with $s/L \ll 1$ and $x = O(s)$, the chord length L can be replaced by infinity as the upper limit of the ξ-integral. This simplification is equivalent to stretching the coordinates, keeping x/s and y/s as new variables and considering the limit $L/s \to \infty$. However, this step alone does not significantly simplify (1), and an additional assumption is required before analytical results can be obtained*. We shall assume, therefore, that the spanwise load distribution is elliptical, as it is known to be in the first-order solution (6). Thus, we replace (6) by the assumed dipole moment

$$\tau(x, y) = 2U\alpha[1 - f(x)](s^2 - y^2)^{\frac{1}{2}} \tag{10}$$

where the new unknown $f(x)$ satisfies the conditions

$$f(x) \to 0 \qquad \text{as } x \to \infty$$
$$f(x) \to 1 + O(x^{\frac{1}{2}}) \qquad \text{as } x \to 0$$

Substitution in (1), with new nondimensional variables $x = sv$, $y = sw$, $\xi = st$, $\eta = su$, and the upper limit L/s replaced by infinity, gives the equation

$$1 = \frac{1}{2\pi} \int_{-1}^{1} du \int_0^\infty dt f'(t) \frac{u}{(1-u^2)^{\frac{1}{2}}} \left\{ \frac{[(v-t)^2 + (w-u)^2]^{\frac{1}{2}}}{(v-t)(w-u)} - \frac{1}{w-u} \right\}$$
$$v>0, \quad |w|<1 \tag{11}$$

Setting $w=0$, thus satisfying this equation on the centerline of the foil, and performing the u-integration, we obtain the one-dimensional integral equation

* Alternatively, one could generate numerical solutions of (1), and examine the limit $L/s > \infty$. One may anticipate that this procedure will be more difficult to carry out than the finite-aspect-ratio computations which are now fairly common.

$$1 = -\frac{1}{\pi}\int_0^\infty dt f'(t)\left\{\frac{[1+(v-t)^2]^{\frac{1}{2}}}{v-t}E([1+(v-t)^2]^{-\frac{1}{2}}) + \frac{\pi}{2}\right\} \quad v>0 \tag{12}$$

where E denotes the complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} d\theta\,(1-k^2\sin^2\theta)^{\frac{1}{2}}\,.$$

The integral equation (12) was derived by Wieghardt [8], who likewise assumed the spanwise loading to be elliptical. Wieghardt gave, as an approximate solution for the bound vortex distribution,

$$\gamma(v) = \frac{\partial\tau}{\partial v} = \frac{1}{2}\frac{1.12}{v^{\frac{1}{2}}+2.4v^3}\,; \qquad 0.1<v<\infty \tag{13}$$

These coefficients appear to have been found numerically, and Wieghardt states that this approximation is not valid near the leading edge. (We have added a factor of $\frac{1}{2}$; Wieghardt does not define his nondimensional γ, and the factor of $\frac{1}{2}$ is consistent with his subsequent calculation of the lift coefficient.)

3. A local solution of the integral equation

Equation (12) is an integral equation of the Wiener–Hopf type, which can be solved, in principle, using Fourier transforms. We shall use that approach here, and while we are unable to solve the full equation for the unknown $f(v)$, we can extract the local behavior near the leading edge, where

$$f(v) \cong 1 - Cv^{\frac{1}{2}} \tag{14}$$

and determine the constant C, or the strength of the leading-edge singularity.

Before proceeding with this task, equation (12) will be put in a more convenient form, by adding to both sides the quantity

$$-\frac{1}{\pi}\int_0^\infty dt f'(t)\left\{\frac{\pi}{2} + \frac{\pi}{2}\operatorname{sgn}(v-t)\right\} = -\int_0^v dt f'(t) = 1 - f(v)$$

(Note that $f(0)=1$.) It follows that (12) is equivalent to the equation

$$f(v) = -\frac{1}{\pi}\int_0^\infty dt f'(t)\,k(v-t)\,, \qquad v>0 \tag{15}$$

where

$$k(u) = \left\{(1+u^2)^{\frac{1}{2}}E\left(\frac{1}{(1+u^2)^{\frac{1}{2}}}\right) - \frac{\pi}{2}|u|\right\}u^{-1}\,. \tag{16}$$

An analogous problem occurs in the lifting-line theory of aerodynamics for foils of high aspect ratio. In that case (15) holds for the bound vorticity of a semi-infinite rectangular wing, but with the simpler kernel $k(u)=u^{-1}$. Stewartson [4] has used the Wiener–Hopf technique in this connection, in an attempt to analyze the local flow near the tip. Stewartson found in that case that (14) held, with

$$C = 2\pi^{-\frac{1}{2}} = 1.12\,.$$

This result is in striking agreement with Wieghardt's approximation (13). Indeed, we could surmise that Stewartson's solution would be valid locally near $x=0$, for our integral equation and kernel (15–16), since the kernel (16) is singular, as $u\to 0$, in precisely the same manner as the lifting-line kernel. However, we know of no general theorem which allows this conclusion to be drawn directly, and thus it is necessary to repeat Stewartson's analysis, with the more complicated kernel (16), utilizing the Wiener–Hopf technique for solving integral equations of the form (15).

First, we replace the kernel $k(u)$ by

$$k(u,\varepsilon)=\varepsilon\left\{(1+u^2)^{\frac{1}{2}}E\left(\frac{1}{(1+u^2)^{\frac{1}{2}}}\right)-\frac{\pi}{2}|u|\right\}K_1(\varepsilon|u|)\,\mathrm{sgn}(u) \tag{17}$$

and note that

$$k(u)=\lim_{\varepsilon\to 0}k(u,\varepsilon).$$

Next we define the one-sided Fourier transforms

$$F_+(\omega)=\int_0^\infty f(v)e^{i\omega v}dv \tag{18}$$

$$M_-(\omega)=\int_{-\infty}^0\left\{-\frac{1}{\pi}\int_0^\infty f'(t)k(v-t,\varepsilon)\right\}e^{i\omega v}dv \tag{19}$$

and the inverse transform

$$f(v)=\frac{1}{2\pi}\int_{-\infty}^\infty F_+(\omega)e^{-i\omega v}d\omega\,,\qquad v>0 \tag{20}$$

$$f(v)=0\,,\qquad v<0 \tag{21}$$

The functions $F_+(\omega)$ and $M_-(\omega)$ are analytic, respectively, in the upper and lower half planes, or for $\mathrm{Im}(\omega)\gtrless 0$. The original integral equation is then equivalent (in the limit $\varepsilon\to 0$) to the equation

$$F_+(\omega)+M_-(\omega)=-\frac{1}{\pi}\int_{-\infty}^\infty e^{i\omega v}\int_0^\infty f'(t)k(v-t),\ \varepsilon)\,dt\,dv\,. \tag{22}$$

The t-integration may be extended to the range $(-\infty,\infty)$, since $f\equiv 0$ for negative t. Then, from the Faltung theorem of Fourier theory,

$$\begin{aligned}F_+(\omega)+M_-(\omega)&=-\frac{1}{\pi}K(\omega)\int_0^\infty f'(v)e^{i\omega v}dv\\&=-\frac{1}{\pi}K(\omega)\{-i\omega F_+(\omega)-f(0)\}\,.\end{aligned} \tag{23}$$

Since $f(0)=1$, it follows that

$$F_+(\omega)\left\{1-\frac{i\omega}{\pi}K(\omega)\right\}=\frac{1}{\pi}K(\omega)-M_-(\omega) \tag{24}$$

Here

$$K(\omega)=\int_{-\infty}^\infty e^{i\omega v}k(v,\varepsilon)\,dv \tag{25}$$

is the Fourier transform of the kernel function. We note, from the definition of $k(v,\varepsilon)$ as an odd function of v, that $K(\omega)$ is an odd function of ω, and $K(0)=0$. For large ω, an asymptotic expression can be found by noting first that, for small v,

$$k(v,\varepsilon)=\frac{1}{v}-\frac{\pi}{2}\,\mathrm{sgn}(v)+O(v)\,. \tag{26}$$

Thus it follows that, for large ω,

$$K(\omega)=\pi i\,\mathrm{sgn}(\omega)-\pi i/\omega+O(\omega^{-2}) \tag{27}$$

(cf. Lighthill, [2]).

Assume the factorization

$$\frac{L_+(\omega)}{L_-(\omega)} = 1 - \frac{i\omega}{\pi} K(\omega) \tag{28}$$

with $L_\pm(\omega)$ analytic and nonzero, respectively, for $\text{Im}(\omega) \gtrless 0$. We then obtain the Wiener–Hopf equation

$$F_+(\omega)L_+(\omega) = \frac{i}{\omega}[L_+(\omega)-L_-(\omega)]-L_-(\omega)M_-(\omega) \tag{29}$$

or

$$F_+(\omega)L_+(\omega) - \frac{i}{\omega}[L_+(\omega)-L_+(0)] = -\frac{i}{\omega}[L_-(\omega)-L_-(0)]-L_-(\omega)M_-(\omega). \tag{30}$$

Assuming that $F_+(\omega)$ is analytic for $\text{Im}(\omega) > -\varepsilon$, and $F_-(\omega)$ is analytic for $\text{Im}(\omega) < \varepsilon$, it follows that both sides of (30) must be equal to a function which is analytic everywhere. To evaluate this function (which, in fact, is zero), we must find the limiting forms of $L_+(\omega)$ and $F_+(\omega)$ as $\omega \to \infty$.

The factorization functions $L_\pm$ can be evaluated by contour integration, since from Cauchy's theorem

$$\log L_+(\omega) - \log L_-(\omega) = \frac{1}{2\pi i}\int_C \log\left\{1 - \frac{i\zeta}{\pi}K(\zeta)\right\}\frac{d\zeta}{\zeta-\omega} \tag{31}$$

if ω is a point within the closed contour C. The contour C can be extended to $\pm$ infinity, on or near the real ζ-axis, and hence separate equations are found for $L_\pm$ in terms of the integrals

$$\log L_\pm(\omega) = \frac{1}{2\pi i}\int_{-\infty\mp i\mu}^{\infty\mp i\mu} \log\left[1 - \frac{i\zeta}{\pi}K(\zeta)\right]\frac{d\zeta}{\zeta-\omega} \tag{32}$$

where $0 < \mu < \varepsilon$. These integrals converge at infinity in the principle-value sense, since $\zeta K(\zeta)$ is an even function for real ζ, and it follows that, for real ω,

$$\begin{aligned}\log L_+(\omega) &= \frac{1}{2\pi i}\int_{-\infty C_-}^{\infty} \log\left[1 - \frac{i}{\pi}\zeta K(\zeta)\right]\frac{d\zeta}{\zeta-\omega} \\ &= \frac{\omega}{\pi i}\int_{0C_-}^{\infty} \log\left[1 - \frac{i}{\pi}\zeta K(\zeta)\right]\frac{d\zeta}{\zeta^2-\omega^2}\end{aligned} \tag{33}$$

where C_- denotes a contour passing below the pole. It is now apparent that

$$\log L_+(0) = 0$$

and thus

$$L_+(0) = 1\,.$$

In order to determine the form of $L_+(\omega)$ as $\omega \to +\infty$, let $\zeta = \omega\eta$. Then

$$\log L_+(\omega) = \frac{1}{\pi i}\int_{0C_-}^{\infty} \log\left[1 - \frac{i}{\pi}\,\omega\eta\, K(\omega\eta)\right]\frac{d\eta}{\eta^2-1}\,. \tag{34}$$

Using the asymptotic expression (27), it follows that

$$\begin{aligned}\log L_+(\omega) &\cong \frac{1}{\pi i}\int_{0C_-}^{\infty} \log(\omega\eta)\frac{d\eta}{\eta^2-1} = \frac{1}{\pi i}\int_0^{-i\infty}\log(\omega\eta)\frac{d\eta}{\eta^2-1} \\ &= \frac{1}{\pi}\int_0^\infty \log(-i\omega u)\frac{du}{u^2+1} = \tfrac{1}{2}\log(-i\omega)\,.\end{aligned}$$

Hence

$$L_+(\omega) = \mathrm{e}^{-\pi i/4}\omega^{\frac{1}{2}} + O(1) \tag{35}$$

as $\omega \to +\infty$. (Note that the dependence on ε has been lost at this point, essentially because ω/ε has been assumed large.) Since L_+ is analytic in the upper-half plane by definition, equation (35) can be extended analytically, into the entire complex ω-plane, with a branch cut on the negative imaginary axis.

The required asymptotic expression for $F_+(\omega)$ is

$$F_+(\omega) = i/\omega f(0) = i/\omega\,. \tag{36}$$

Thus it follows that the left side of the Wiener–Hopf equation (30) vanishes at infinity, and hence is identically zero. Solving (30) for F_+, we then obtain

$$\begin{aligned} F_+(\omega) &= \frac{i}{\omega}\left[1 - \frac{1}{L_+(\omega)}\right] \\ &= \frac{i}{\omega} - ie^{\pi i/4}\omega^{-\frac{3}{2}} + O(\omega^{-2})\,. \end{aligned} \tag{37}$$

From the inverse transform (20), the solution of the integral equation (15) is

$$\begin{aligned} f(v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i}{\omega}\left[1 - \frac{1}{L_+(\omega)}\right] e^{-i\omega v} d\omega \\ &\cong \frac{1}{2\pi}\int_{-\infty C_-}^{\infty} \left[\frac{i}{\omega} - ie^{\pi i/4}\omega^{-\frac{3}{2}} + \dots\right] e^{-i\omega v} d\omega \end{aligned} \tag{38}$$

where the contour is deformed into the upper half plane, before expanding the function F_+, since this function is analytic in $\mathrm{Im}(\omega) > 0$. Term-by-term integration is then justified, and the first term is evaluated by residue theory as

$$\frac{1}{2\pi}\int_{-\infty C_+}^{\infty} \frac{i}{\omega} e^{-i\omega v} d\omega = 1 \tag{39}$$

since the remaining contour integral, in the lower half plane, can be deformed to $-i\infty$ and hence vanishes. To evaluate the second term in (38), we first integrate by parts to obtain

$$-\frac{v}{\pi} e^{\pi i/4}\int_{-\infty C_+}^{\infty} \omega^{-\frac{1}{2}} e^{-i\omega v} d\omega\,. \tag{40}$$

The contour is then deformed into the lower half plane, but passing above the branch-cut along the negative imaginary axis. The last expression becomes

$$-\frac{2iv}{\pi}\int_0^{-i\infty} e^{-i\omega v}\frac{d\omega}{|\omega|^{\frac{1}{2}}} = -2(v/\pi)^{\frac{1}{2}}\,.$$

Thus it follows that, near $v=0$,

$$f(v) = 1 - 2\pi^{-\frac{1}{2}} v^{\frac{1}{2}} + O(v)\,. \tag{41}$$

This is the desired leading-edge approximation, which establishes the value of the coefficient C in equation (14).

4. The leading-edge suction force

A check on the preceding analysis can be made by computing the suction force acting on the leading edge. It is known that a rectangular wing of low-aspect ratio will experience lift and drag forces (cf. Robinson and Laurmann, [3], pp. 275–277):

$$L = \pi\rho U^2 s^2 \alpha \tag{42}$$

$$D = \tfrac{1}{2} L\alpha \tag{43}$$

Thus there exists a leading-edge suction force of magnitude

$$-\tfrac{1}{2}L\alpha = -\frac{\pi}{2}\rho U^2 s^2 \alpha^2 \tag{44}$$

(It should be noted that some questions exist regarding the validity of these results, for rectangular foils, and indeed the existence of a leading-edge suction force has been questioned in this case (cf. Ward, [7], p. 210). We take the view here that the force coefficients (42) and (43) are valid, since they can be derived asymptotically, without regard for the local leading-edge behavior (the drag being computed from the kinetic energy in the wake); thus, if the local discrepancy at the leading edge is confined to a chordwise interval which is $O(s)$, its effect on the lift and drag forces must decrease uniformly as the aspect ratio $2s/L$ decreases to zero. With respect to the leading-edge suction force, we note that, if there exists a finite span $2s$, and a (nonzero) square-root local flow at the leading edge, there must be an associated leading-edge suction force.)

To compare with the above results, we shall compute the leading-edge suction force directly. By combining equations (2, 10, 41), it follows that the velocity potential near the leading edge of a rectangular flat plate is of the form

$$\phi(x, y, 0\pm) = \pm 2U\alpha\pi^{-\frac{1}{2}}(s^2-y^2)^{\frac{1}{2}}(x/s)^{\frac{1}{2}}, \qquad x \ll s. \tag{45}$$

If $x+iz=\mathrm{R}e^{i\theta}$ is a complex variable, the complex velocity corresponding to (45) is

$$\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial z} = U\alpha\pi^{-\frac{1}{2}}s^{-1}(s^2-y^2)^{\frac{1}{2}}(R/s)^{-\frac{1}{2}}e^{-i\theta/2}, \qquad R \ll s \tag{46}$$

and from Blasius' theorem the leading-edge suction force is

$$X = \mathrm{Re}-\tfrac{1}{2}\rho \int_0^{2\pi} \left(\frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial z}\right)^2 \mathrm{R}e^{i\theta}\,d\theta$$
$$= -\rho U^2\alpha^2(s^2-y^2)/s. \tag{47}$$

Integrating along the leading edge in the range $-s<y<s$, we obtain the total suction force

$$\bar{X} = -\tfrac{4}{3}\rho U^2\alpha^2 s^2. \tag{48}$$

Comparison with the "exact" result (44) reveals a relative discrepancy of

$$\frac{8}{3\pi} \doteq 0.85, \tag{49}$$

or a 15% error in the integrated leading-edge singularity. It seems likely that this discrepancy results from the assumption of an elliptic spanwise load distribution, which is clearly in error near the corners. We could correct the local solution (45) by a multiplicative correction factor of $(3\pi/8)$, but on the assumption that the greatest source of error is indeed at the corners, it seems preferable to leave (45) unchanged.

5. Local flow near the ship's bow

We shall now apply the local leading-edge solution to the case of a ship's bow, with the assumptions stated in the Introduction. Thus s is the draft of the ship, α is the yaw angle, the stem is assumed to be vertical, and the free-surface boundary condition is replaced by the simpler rigid-wall condition $\phi_y(x, 0, z)=0$. (Note that in this case y is the vertical axis, coincident with the ship's bow.)

To correct for finite thickness, and hence remove the square-root infinity which follows from direct application of (45) or (46), we apply Lighthill's rule (cf. Van Dyke, [6]). Thus, if $r(y)$ denotes the radius of curvature of the stem, a uniformly-valid correction to (45) is

$$\phi = \pm 2U\alpha\pi^{-\frac{1}{2}}(s^2-y^2)^{\frac{1}{2}}(x-\tfrac{1}{2}r)^{\frac{1}{2}}s^{-\frac{1}{2}}, \tag{50}$$

and the two-dimensional complex velocity in the horizontal plane may be written

$$\phi_x - i\phi_z = U\alpha\pi^{-\frac{1}{2}}(s^2-y^2)^{\frac{1}{2}}(x+iz-\tfrac{1}{2}r)^{-\frac{1}{2}}s^{-\frac{1}{2}}. \tag{51}$$

The maximum value of the tangential velocity occurs on the forward edge of the stem, at which point

$$v_t = U\alpha\left(\frac{2}{\pi rs}\right)^{\frac{1}{2}}(s^2-y^2)^{\frac{1}{2}}. \tag{52}$$

We note that (51) includes only the perturbation velocity associated with the yawed inflow velocity vector. To this must be added the inflow velocity $(U, 0, \alpha U)$ and the perturbation associated with the symmetrical thickness problem. The latter will depend, in general, on the entire thickness distribution over the hull surface, but for the bow portion alone it will be of order r and, generally speaking, (51) should dominate the other effects if the yaw angle α is nonzero and the stem radius r is reasonably small.

As a rough practical calculation, let $\alpha = 5°$ (10^{-1} radians), $s = 30$ feet, and $r = 0.5$ feet; then from (52), with $y = 0$,

$$v_t/U \doteq 0.62.$$

This factor will diminish, slowly, with increasing depth.

Another feature of practical interest is the vertical component of induced velocity associated with oblique inflow, or the spanwise component associated with tip roll-off in the aerodynamic situation. In fact, the first-order solution (6) can be utilized for this purpose, without correcting for leading-edge effects, with the predicted spanwise velocity component

$$\frac{\partial\phi}{\partial y} = \mp U\alpha y(s^2-y^2)^{-\frac{1}{2}} \quad \text{on} \quad z = 0\pm, \qquad x \gg s.$$

Correcting for the leading-edge effect, we find from (45) the vertical velocity component

$$\frac{\partial\phi}{\partial y} = \mp 2U\alpha\pi^{-\frac{1}{2}}y(s^2-y^2)^{-\frac{1}{2}}(x/s)^{\frac{1}{2}} \quad \text{on} \quad z = 0\pm, \qquad x \ll s.$$

This prediction is invalid near the corner, or the intersection of the ship's bow and keel, and to correct for this deficiency would require us to examine the lifting-surface problem anew, for a foil of infinite chord length and semi-infinite span. Such an effort would be of doubtful value since the planform of practical ships is not strictly rectangular. Moreover, the domains of validity of the two expressions above do not overlap, and to overcome this defect with a complete solution, valid when $x = O(s)$, would require the determination of the complete "inner solution" or the complete solution of the Wiener–Hopf one-dimensional integral equation (15). It does not seem likely that this task can be accomplished without resort to a numerical solution.

The neglect of free-surface effects may be a more important deficiency of our results. Hu [1] has shown that for a low-aspect-ratio rectangular lifting surface, situated vertically in the free surface and moving with steady forward velocity, the first-order solution of the full problem in terms of small-aspect-ratio is governed by rigid free-surface condition. Indeed this important result can be inferred from the linear free-surface condition

$$U^2\frac{\partial^2\phi}{\partial x^2} + g\frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0$$

when one recalls that, in the small-aspect-ratio theory, gradients with respect to the longitudinal coordinate x are negligible compared to the lateral and vertical coordinates y and z. This argument is not valid, however, in the vicinity of the bow or leading edge, and it would be desirable to examine this problem more closely.

Finally, there is the simpler alternative of an experimental investigation. For this purpose a standard ship model could be towed at small yaw angles, and a velocity survey made in the vicinity of the bow, but boundary-layer thicknesses comparable to the stem radius may complicate this task, and it would be necessary to increase the stem radius, hopefully obtaining an experimental geometry which is intermediate between the pitfalls of viscous effects (when the radius is too small), and dominance of other velocity components, such as the free stream and thickness effect of the hull, when the radius is too large. Such an experimental investigation, if successful, would add substantially to the value of the present theory.

REFERENCES

[1] P. Hu, Forward speed effect on lateral stability derivatives of a ship, *Davidson Laboratory Report No.* 829 (1961).
[2] M. J. Lighthill, *An Introduction to Fourier Analysis and Generalized Functions*, Cambridge University Press (1958).
[3] A. Robinson and J. A. Laurmann, *Wing Theory*, Cambridge University Press (1956).
[4] K. Stewartson, A note on lifting line theory, *Quarterly Journal of Mechanics and Applied Mathematics*, 13, 1 (1960) 49–56.
[5] M. Van Dyke, Lifting-line theory as a singular-perturbation problem, *Proceedings of the Sixth Symposium on Fluid Dynamics, Zakopane, Poland* (1963) 369–382, Oxford: Pergamon Press.
[6] M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, New York, Academic Press (1964).
[7] G. N. Ward, *Linearized Theory of Steady High-Speed Flow*, Cambridge University Press (1955).
[8] Karl Wieghardt, Chordwise load distribution of a simple rectangular wing, *Zeitschrift für angewandte Mathematik und Mechanik*, 19, 5 (1939) 257–270. (English translation: *N.A.C.A., T.M. No.* 963.)